

Browser-Native Prandtl–Meyer Expansion: The Mach Number That Was the Edge of the Search

JJ Ben-Joseph^{1,*}

¹TensorSpace, Inc, Tel Aviv, Israel

*jjbj@tensorspace.ai

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Interactive calculator: <https://agentcalc.com/prandtl-meyer-expansion-calculator>

Abstract

A centered expansion fan in a calorically perfect gas accelerates a supersonic stream through an angle equal to the change in the Prandtl–Meyer function. That function is strictly increasing, from zero at Mach number 1 to a finite vacuum limit, so a requested angle below the limit has exactly one downstream Mach number and an angle above the limit has none. A browser calculator inverted the function by bisection on the fixed interval $1 \leq M \leq 50$. The value of the function at the upper endpoint, for $\gamma = 1.4$, is 124.729° , which is 5.725° short of the vacuum limit 130.454° . Every target above 124.729° was therefore returned as $M = 50$. From an upstream Mach number of 2, a 100° turn is a real expansion: the root is $M_2 = 70.2873$, and the static-pressure ratio belonging to $M = 50$ is 10.811 times the pressure ratio of that root. A 110° turn has no finite Mach number. The greatest expansion from $M_1 = 2$ is 104.074° , and the same search reported 50 for it. A zero turn at $M_1 = 80$ is the identity. The search reported $M_2 = 50$ and a pressure ratio of 26.729, a compression produced by the bracket. The replacement is Newton’s method on the analytic derivative, started at the upstream Mach number, with the residual $\nu(M) - \nu_2$ reported in the result. Targets at or above the vacuum limit, and negative turns, are refused rather than assigned an endpoint.

1 Introduction

The Prandtl–Meyer fan is the isentropic way a supersonic stream turns a convex corner. The wall angle does not appear as a term in a shock relation. It is added to a function of Mach number, and the downstream Mach number is whatever Mach number has that new value of the function [1–3]. The function is old, closed-form, and strictly increasing. Inverting it is a one-dimensional root problem with a known range.

A root problem with a known range is easy to solve badly by refusing to look outside a convenient interval. The previous AgentCalc page bisected $\nu(M)$ on $1 \leq M \leq 50$ and always returned the final midpoint. If the target lay above $\nu(50)$, every trial inside the interval was too small, the upper bound never moved in spirit, and the number that came back was the endpoint $M = 50$. The page then converted that Mach number into isentropic pressure, temperature, and density ratios and displayed them with three decimals, in the same sentence it would have used for a genuine root.

The endpoint is not a rare curiosity. For $\gamma = 1.4$, $\nu(50) = 124.729^\circ$ and the vacuum limit is $\nu_{\max} = 130.454^\circ$. The 5.725° between those two angles contains every expansion whose true Mach number exceeds 50, and everything beyond $\nu_{\max}$ contains the expansions that do not exist. Both

were labeled $M_2 = 50$. A separate failure sits at the other end of the same bug. An upstream Mach number above 50, turned through zero degrees, has downstream Mach number equal to itself. The bisection cannot return a number above 50, so a zero turn at $M_1 = 80$ came back as $M_2 = 50$ and a pressure rise of 26.729. The bracket had invented a compression.

This note documents the [Prandtl–Meyer expansion calculator](#) after that search was replaced. The downstream Mach number is a Newton root of $\nu(M) - \nu_2 = 0$ when the target lies in the open interval between 0 and $\nu_{\max}$. The residual, the iteration count, and the three angles $\nu(M_1)$, ν_2 , and $\nu_{\max}$ are part of the result. A target outside that interval is reported as having no finite Mach number, together with the greatest turn the upstream state can still support. The benchmarks below are the 5° expansion the form opens on, the 100° expansion whose root lies above 50, the 110° turn that has no root, and the zero turn at $M_1 = 80$.

2 Model

2.1 The fan and the function

Meyer’s 1908 dissertation treated the steady two-dimensional motion of a gas at supersonic speed, including the centered expansion at a convex corner [1]. In a calorically perfect gas the turning is measured by the Prandtl–Meyer function. One standard closed form, the form evaluated on this page and tabulated for air in NACA Report 1135, is

$$\nu(M) = \sqrt{\frac{\gamma+1}{\gamma-1}} \arctan\left(\sqrt{\frac{\gamma-1}{\gamma+1}}(M^2-1)\right) - \arctan\sqrt{M^2-1}, \quad (1)$$

with ν in radians and $\gamma > 1$ the constant heat-capacity ratio [4, 5]. The same report records the differential relation from which (1) is integrated,

$$\frac{d\nu}{dM} = \frac{\sqrt{M^2-1}}{M\left(1 + \frac{1}{2}(\gamma-1)M^2\right)}. \quad (2)$$

For every $M > 1$ the numerator and the denominator are positive, so ν is strictly increasing. It starts at $\nu(1^+) = 0$. It cannot grow forever: as $M \rightarrow \infty$ the two arctangents in (1) each approach $\pi/2$, and

$$\nu_{\max} = \frac{\pi}{2} \left(\sqrt{\frac{\gamma+1}{\gamma-1}} - 1 \right). \quad (3)$$

For $\gamma = 1.4$, $\nu_{\max} = 130.454^\circ$. For a monatomic gas, $\gamma = 5/3$, the square root equals 2 and $\nu_{\max} = \pi/2$, which is 90° exactly. A requested value inside $(0, \nu_{\max})$ is attained once. A requested value at or above $\nu_{\max}$ is not a Mach number.

Across a simple expansion wave the Mach number and the flow angle change together [2, 3]. For a centered fan that turns the wall through a positive angle θ ,

$$\nu(M_2) = \nu(M_1) + \theta, \quad (4)$$

where θ and ν are in the same unit. The page takes θ in degrees and converts it before the addition. The downstream Mach number is the inverse of (1) at the right-hand side of (4). Because of (2), that inverse is unique when it exists.

2.2 Isentropic ratios

A Prandtl–Meyer fan is isentropic. The stagnation temperature and the stagnation pressure are the same on both sides, so the static temperature ratio between two Mach numbers is

$$\frac{T_2}{T_1} = \frac{1 + \frac{1}{2}(\gamma - 1)M_1^2}{1 + \frac{1}{2}(\gamma - 1)M_2^2}, \quad (5)$$

and the static pressure ratio of a calorically perfect gas is

$$\frac{p_2}{p_1} = \left(\frac{T_2}{T_1} \right)^{\gamma/(\gamma-1)}. \quad (6)$$

The density ratio is $(p_2/p_1)/(T_2/T_1)$. These are the ratios tabulated, as functions of Mach number, for continuous perfect-gas flow in NACA Report 1135, which superseded the Ames wind-tunnel tables of 1947 [4, 6]. They are not shock relations. They move with M_2 : a larger downstream Mach number is a cooler, lower-pressure, less dense state. An erroneous M_2 moves all three of them.

Evaluated at $M = 2$ and $\gamma = 1.4$, (1) is 26.37976° . Rounded to three decimals, that is 26.380° , the value given for this Mach number in the perfect-gas Prandtl–Meyer table of NACA Report 1135. The calculator is using the function that table records. The failures below are failures of the inversion, not of the closed form.

3 What the page returns

The form takes $M_1 > 1$, a turn angle in degrees, and $\gamma > 1$. The default is the worked example: $M_1 = 2$, $\theta = 5^\circ$, $\gamma = 1.4$. A negative angle is rejected. It is a compression corner, and this page does not solve an oblique shock. Equal inputs with $\theta = 0$ are an identity: $M_2 = M_1$, every static ratio is 1, and the residual is zero because $\nu(M_1)$ is already the target.

For a positive turn the page forms $\nu_2 = \nu(M_1) + \theta$ and compares it with (3). If $\nu_2 \geq \nu_{\max}$, the result is that no finite Mach number exists, together with the greatest expansion still available, $\nu_{\max} - \nu(M_1)$. Otherwise Newton’s method solves $\nu(M) - \nu_2 = 0$, starting at M_1 . The update is (2). A step that would leave the supersonic axis is halved. The iteration stops when the absolute residual is below $10^{-12} \max(1, |\nu_2|)$, and it gives up after 40 iterations or if the iterate exceeds 10^{12} . On the cases in this note it does not get past ten iterations, and the 5° turn finishes in four.

The displayed result contains M_2 , the residual converted to degrees, the iteration count, $\nu(M_1)$, ν_2 , $\nu_{\max}$, the greatest remaining expansion, and the three isentropic ratios. A Mach number is not shown unless it is a root of (4). The expansion-fan mini-game on the same HTML page is a separate angle-matching animation. It does not call this inverse, and it is not part of the released science-page copy.

4 Four states

Table 1 uses $\gamma = 1.4$. The legacy column is the old bisection on $[1, 50]$, one hundred steps, returning the final midpoint whether or not the target was inside ν of that interval.

The 5° column is the case the form opens on, and it is the case the old search happened to contain. Here $\nu(M_1) = 26.3798^\circ$ and $\nu_2 = 31.3798^\circ$, well below $\nu(50) = 124.729^\circ$. Four Newton steps from $M_1 = 2$ reach $M_2 = 2.186428$, and the residual is -1.11×10^{-16} radians. The temperature ratio is 0.920201 and the density ratio is 0.812283. One hundred bisections on $[1, 50]$ reproduce this

Table 1: Downstream states for $\gamma = 1.4$. The legacy column is bisection on $1 \leq M \leq 50$. The pressure factor is the legacy static-pressure ratio divided by the pressure ratio of the Newton root. A dash means the legacy value is not a root.

	$M_1 = 2, 5^\circ$	$M_1 = 2, 100^\circ$	$M_1 = 2, 110^\circ$	$M_1 = 80, 0^\circ$
Newton M_2	2.18643	70.2873	none	80
Legacy M_2	2.18643	50	50	50
Newton iterations	4	10	—	0
Residual (rad)	-1.11×10^{-16}	4.44×10^{-16}	—	0
p_2/p_1 , root	0.747464	2.571×10^{-10}	—	1
p_2/p_1 , legacy	0.747464	2.780×10^{-9}	2.780×10^{-9}	26.729
Pressure factor	1	10.811	—	26.729

Mach number to far more than the three decimals the old page printed. On a turn the bracket was built to hold, the two solvers agree. The bug is not visible here. That is why a single modest example cannot audit the inverse.

The 100° column is the same upstream state with a turn the bracket cannot hold. The target is $\nu_2 = 126.380^\circ$. That is 1.651° above $\nu(50)$ and 4.074° below $\nu_{\max}$, so a finite root exists and it is greater than 50. Newton, still started at $M_1 = 2$, reaches $M_2 = 70.28729$ in ten iterations with a residual of 4.44×10^{-16} radians. The old bisection returns 50 exactly. Equation (6) then turns the two Mach numbers into static-pressure ratios 2.571×10^{-10} and 2.780×10^{-9} . The endpoint overstates the pressure ratio by a factor of 10.811. The temperature ratio is overstated by a factor of 1.974. Both numbers were displayed as the expansion. The search and the root part company at a turn of 98.349° from this upstream state, which is the angle that makes $\nu_2 = \nu(50)$. Every larger finite expansion was being reported as the same Mach number.

The 110° column has no root. The target is 136.380° , which exceeds $\nu_{\max} = 130.454^\circ$ by 5.926° . The greatest expansion from $M_1 = 2$ is $\nu_{\max} - \nu(M_1) = 104.074^\circ$. The old search does not have a way to say this. Its comparisons all fall on the side that raises the lower bound, and it returns 50, with the same pressure ratio 2.780×10^{-9} it returned for the genuine 70.287 root's neighbor. One displayed Mach number was doing duty for a real supersonic state and for a turn into a vacuum the function does not attain.

The last column is a zero turn, which should be the easiest case on the page. At $M_1 = 80$, $\nu(M_1)$ is already the target, the residual at the starting guess is zero, and the downstream state is $M_2 = 80$ with every static ratio equal to 1. The greatest further expansion from this Mach number is only 3.580° , because 80 is already close to the vacuum limit and 50 is not. The old bisection, forbidden to return 80, returns 50. Equation (6) at those two Mach numbers is a pressure ratio of 26.729 and a temperature ratio of 2.557. A calculator asked for no turn at all described a strong isentropic compression. The sign of the error is the opposite of the 100° case, and the cause is the same endpoint.

5 The function and the clipped inverse

Figure 1 draws the same statements for $\gamma = 1.4$. Panel (a) is $\nu(M)$ from (1). The curve leaves zero just above $M = 1$, passes 26.380° at $M = 2$, and flattens toward $\nu_{\max} = 130.454^\circ$. The lower horizontal line is $\nu(50) = 124.729^\circ$. Everything between that line and the vacuum limit was inside the range of the function and outside the range of the old search. Panel (b) is the downstream

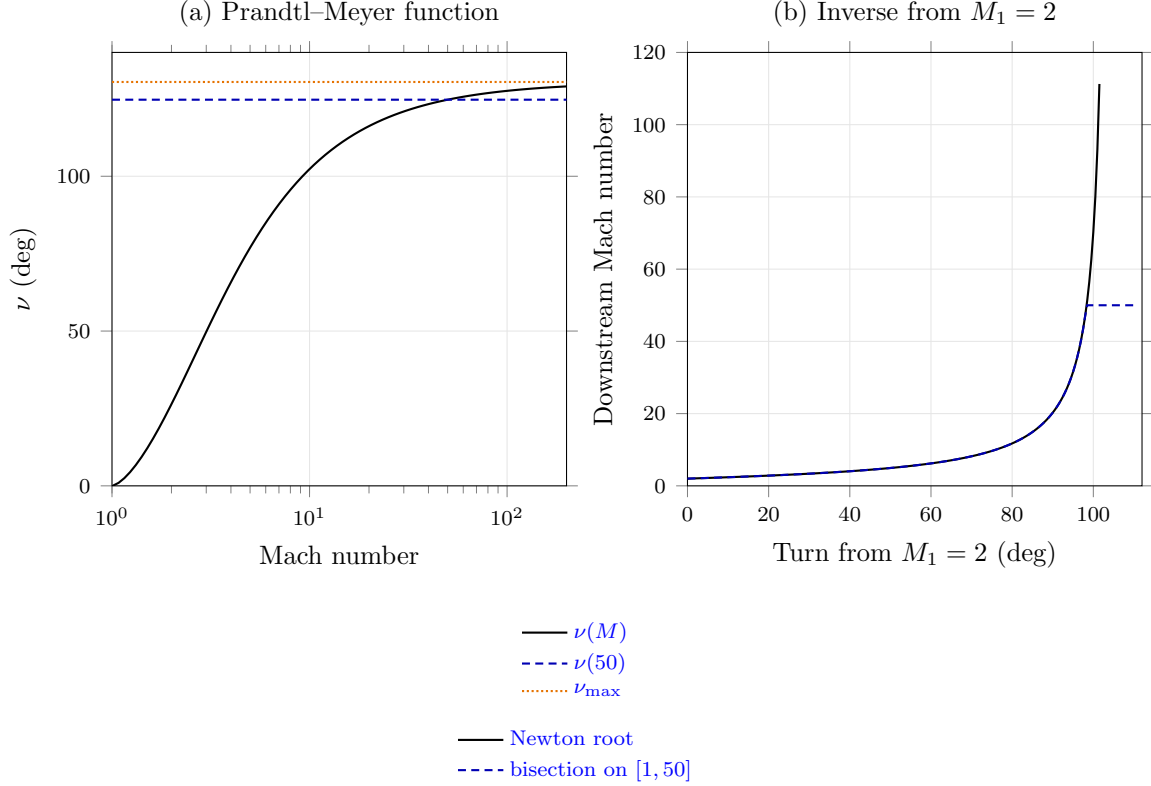


Figure 1: Calorically perfect gas at $\gamma = 1.4$. Panel (a): the Prandtl-Meyer function, the value attained at $M = 50$, and the vacuum limit. Panel (b): downstream Mach number for an expansion from $M_1 = 2$. The Newton root is plotted while it remains at or below 120. The dashed curve is the old bisection, which follows the root until a turn of 98.349° and then remains at 50, including past the greatest finite expansion at 104.074° .

Mach number for an expansion from $M_1 = 2$. Up to a turn of 98.349° the Newton root and the old bisection lie together. Past that turn the old result sits at 50 through the greatest physical turn, 104.074° , and on through 110° , where no root exists. The Newton curve is drawn only while $M_2 \leq 120$, so that the split at 100° ($M_2 = 70.287$) is readable. By 104° the root is already $M_2 = 3854.87$, which is a mathematical expansion within 0.074° of the vacuum limit and not a Mach number one would use for a perfect-gas design. The figure’s job is to show where the endpoint and the root separate, not to suggest that the far end of the curve is an engineering state.

6 Discussion

The numerical claim is narrower than “Newton is the right way to invert a supersonic expansion,” and the narrower claim is the one Table 1 supports. On the 5° turn the old interval contains the root, and a hundred bisections agree with Newton through the three decimals the old page showed. The old interval’s distinctive behavior is outside that turn. It assigns $M = 50$ to a target it has not attained, and it assigns $M = 50$ to a target the function cannot attain, and the calling code cannot tell those two outcomes apart because both arrive as an ordinary floating-point Mach number. The pressure ratio is then an honest evaluation of (6) at a dishonest Mach number. A factor of 10.811

in pressure, or a pressure ratio of 26.729 on a zero turn, is what that composition looks like.

Reporting the residual separates the cases the endpoint had glued together. After four or ten iterations the residual on a genuine root is a unit roundoff in (1), on the order of 10^{-16} radians. There is no residual to report for the 110° turn, because there is no candidate Mach number. The sentence in the result is the comparison with $\nu_{\max}$ instead. That is a different kind of answer, and it should look different on the page. An implementation that wants to keep bisection can do so. The defect was not the choice of bisection. It was a bracket that did not move when the target lay outside it, and a return value that did not say so. Newton’s method is used here because (2) is already the integrand of the function, the upstream Mach number is a natural starting guess, and the iteration count is then a check that the root was cheap. A step that leaves $M > 1$ is halved so a wild iterate cannot become a displayed Mach number by changing sign.

The zero-turn column is the cleanest statement of the same point. An inverse that cannot return the Mach number it was started from, when the angle added is zero, is not an inverse of ν on the supersonic axis. It is an inverse on $[1, 50]$. Those are different functions. They agree on the 5° textbook turn and disagree as soon as the upstream state or the turn leaves that interval. A page whose only worked example sits inside the interval will look correct in its own demonstration and still invent a compression at $M_1 = 80$.

None of this improves the physical model. The residual certifies (4) for the γ the user typed. It does not certify that a laboratory gas at the temperature implied by $M_2 = 70$ still has $\gamma = 1.4$, or that the corner is sharp, or that the fan is free of a downstream shock. NACA Report 1135 already distinguishes the perfect-gas tables from the imperfect-gas corrections [4]. This calculator is the perfect-gas closed form only.

7 Limitations

The implementation is a single centered fan in a steady, two-dimensional, calorically perfect gas of constant γ . Viscosity, heat transfer, a boundary layer on the wall, and a shock that crosses the fan are outside it [3, 5]. So is any change of composition. Near $\nu_{\max}$ the mathematical Mach number becomes enormous while the temperature ratio collapses, and the constant- γ assumption has already failed for air long before $M_2 = 70$. The root is still the right root of the equation the page names. It is not a prediction of a nozzle flow at that Mach number.

A negative turn is refused rather than interpreted as an isentropic compression. A sharp concave corner is an oblique shock, with two branches and a detachment angle, and it is a different calculation. A smooth isentropic compression can be written with (4) and a negative θ until waves coalesce, but this page does not offer that reading. The input is an expansion angle.

The stopping test and the cap at $M = 10^{12}$ are numerical. A turn within a tiny fraction of a degree of $\nu_{\max}$ has a huge, well-defined Mach number. If the iteration were to exceed the cap, the page would say that it did not converge, not that the Mach number equals the cap. The cases in Table 1 do not approach that cap. The 104° turn, 0.074° short of the maximum, converges to $M_2 = 3854.87$ in fifteen iterations. That value is a check on the solver, not a recommended design point, and it is outside the window of Figure 1(b) for that reason.

Report 1135 and the textbooks cited here also treat normal shocks, oblique shocks, and cones [2, 4, 5]. Those relations are not inverted on this page, and agreement of the fan with its own residual says nothing about them.

8 Conclusion

A Prandtl–Meyer expansion is auditable when the Mach number, the residual, and the range of ν are the same statement. The function (1) increases from 0 to $\nu_{\max}$, so $\nu_2 = \nu(M_1) + \theta$ has one supersonic root when $0 < \nu_2 < \nu_{\max}$ and no finite root otherwise. Newton’s method finds the root from M_1 in four iterations for the default 5° turn, where $M_2 = 2.186428$, and in ten iterations for a 100° turn, where $M_2 = 70.2873$. A bisection confined to $M \leq 50$ reports 50 for that turn, and the pressure ratio of the endpoint is 10.811 times the pressure ratio of the root. The same endpoint is what the old search returned for a 110° turn, which exceeds the 104.074° available from $M_1 = 2$, and for a zero turn at $M_1 = 80$, where the downstream state is 80 and the endpoint’s pressure ratio of 26.729 is not a flow. The Mach number 50 was the edge of the search.

Data and code availability

The calculator is the AgentCalc page [prandtl-meyer-expansion-calculator](#), and the released copy is in the [agentcalc.science](#) repository. The benchmark script `generate_prandtl_meyer_figure.py` and the series plotted in Figure 1 are archived with this note. No laboratory expansion was analyzed.

References

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